Lecture no 28

Theorem 1
The First Shift Theorem

The first Shift theorem states that if **L{F(t)}= f(s)** then **L{$e^{(-at)}$ F(t)}= f(s+a)**
The transform **L{$e^{(-at)}$ F(t)}** is thus the same as **L{F(t)}** with s everywhere in the result replaced by (s+a)

Example

L{sin 2t}= $\dfrac{2}{s^2+4}$

Example

then

L{e^{-3t} sin 2t}= $\dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

$L\{t^2\} = \dfrac{2}{s^3}$;

$L\{t^2e^{4t}\}$ is the same with s replaced by (s-4)

So $L\{t^2e^{4t}\} = \dfrac{2}{(s-4)^3}$

Theorem 2 Multiplying by t

If L{F(t)}= f(s) **then**

$$L\{t(F(t)\} = -\frac{d}{ds}\{f(s)\}$$

Example

L{sin 2t}=

$$\frac{2}{s^2+4}$$

And
$$L\{t \sin 2t\} = -\frac{d}{ds}\left[\frac{2}{(s^2+4)}\right] = \frac{4s}{(s^2+4)^2}$$

Example

$$L\{t\cos 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = -\frac{(s^2-9)-s(2s)}{(s^2-9)^2} = -\frac{s^2-9-2s^2}{(s^2-9)^2} = \frac{s^2+9}{(s^2-9)^2}$$

We could, if necessary, take this a stage further and find

$$L\{t^2\cos 3t\} = -\frac{d}{ds}\left(\frac{s^2+9}{(s^2-9)^2}\right) = \frac{2s(s^2+27)}{(s^2-9)^3}$$

Theorem Obviously extends the range of function that we can deal with. So, in general

If L{F(t)}= f(s) **then**

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$$

Theorem Dividing by t

If L{F(t)}= f(s) **then**

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(s)\,ds$$

Example Determine

$$L\left\{\frac{\sin at}{t}\right\}$$

$$\text{As } L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

Example Determine

$$L\left\{\frac{1-\cos 2t}{t}\right\}$$

As $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

Then by Theorem 3,

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty = \left[\frac{1}{2}.2\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \left[\frac{1}{2}\ln s^2 - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln s - \ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{s^2}{(s^2+4)}\right]_s^\infty$$

When $s \to \infty$ *then* $\ln \frac{s^2}{(s^2+4)} \to \ln 1 = 0$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = -\frac{1}{2}\ln \frac{s^2}{(s^2+4)} = \ln \left[\frac{s^2}{(s^2+4)}\right]^{-\frac{1}{2}} = \ln \sqrt{\frac{s^2+4}{s^2}}$$

Standard Forms

F(t)	L {F(t)} = f(s)

a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
sin at	$\frac{a}{s^2+a^2}$
cos at	$\frac{a}{s^2+a^2}$
sinh at	$\frac{a}{s^2-a^2}$
cosh at	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$

(n a positive integer)

Theorem 1 **The First Shift**

Theorem If L{F(t)}= f(s) then **L{$e^{(-at)}$ F(t)}= f(s+a)**

Theorem 2 **Multiplying by t**

If L{F(t)}= f(s) **then**

$$L\{tF(t)\} = -\frac{d}{ds}\{f(s)\}$$

Theorem 3 **Dividing by t**

If L{F(t)}= f(s) **then**

$$L\{\frac{F(t)}{t}\} = \int_s^{\infty} f(s)ds$$

Provided

$$\lim_{t\to 0}\left\{\frac{F(t)}{t}\right\} \textit{ exists.}$$

Inverse Transforms

Here we have the reverse process i.e. given a Laplace transform, we have to find the function of t to which it belongs. For example, we know that $\frac{a}{s^2+a^2}$ is the Laplace Transform of **sin at** , so we can now write $L^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at$, the symbol L^{-1} indicating the inverse transform and not a reciprocal.

(a) $L^{-1}\left\{\frac{1}{s-2}\right\}=e^{-2t}$

(b) $L^{-1}\left\{\frac{s}{s^2+25}\right\}=\cos 5t$

(c) $L^{-1}\left\{\frac{4}{s}\right\}=4$

(d) $L^{-1}\left\{\frac{12}{s^2-9}\right\}=4\sinh 3t$

But what about $L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}$, it happens that we can write $\frac{3s}{s^2-s-6}$ as the sum of two simpler functions $\frac{1}{s+2}+\frac{2}{s-3}$ which , of course , makes all the difference , since we can now proceed.

$$L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}\right\}+L^{-1}\left\{\frac{2}{s-3}\right\}=e^{-2t}+2e^{3t}$$

Rules of Partial Fractions

1. The numerator must be of lower degree than denominator. If it is not , then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fraction.
3. A linear factor (s+a) gives a partial fraction

$$\frac{A}{s+a}$$

is a constant to be determined.

4. A repeated factor $(s+a)^2$ gives

$$\frac{A}{s+a} + \frac{B}{(s+a)^2}$$

5. Similarly $(s+a)^3$ gives

$$\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$$

6. A quadratic Factor (s^2+ps+q) gives $\frac{Ps+Q}{s^2+ps+q}$
7. Repeated quadratic Factor $(s^2+ps+q)^2$ gives

$$\frac{Ps+Q}{s^2+ps+q} + \frac{Rs+T}{(s^2+ps+q)^2}$$

So $\frac{s-19}{(s-2)(s+5)}$ has partial fraction of the form $\frac{A}{(s-2)} + \frac{B}{(s+5)}$

and $\frac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fraction $\frac{A}{(s+3)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$

Example

To determine

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$$

a) First we check that the numerator is of lower degree than the denominator. In fact this is so.
b) Factories the denominator

$$\frac{5s+1}{s^2-s-12} = \frac{5s+1}{(s-4)(s+3)} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

We therefore have an identity

$$\frac{5s+1}{s^2-s-12} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

which is true for any value of s we care to substitute
If we multiply through by the denominator (s^2-s-12) we have

5s + 1 ≡ A(s + 3) + B(s - 4)

We now substitute covenant values for s

i) Let (s -4) =0 that is s = 4 therefore 21 = A(7) + B(0) $\Rightarrow$ A= 3
ii) Let (s + 3) =0 that is s = -3 therefore B = 2

So $\frac{5s+1}{s^2-s-12} = \frac{3}{(s-4)} + \frac{2}{(s+3)}$

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = 3e^{4t} + 2e^{-3t}$$

Example Determined

$$L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$$

$$L\{F(t)\} = \frac{9s-8}{s^2-2s}$$

a) Numerator of first degree ; denominator of second degree.

b) $\dfrac{9s-8}{s^2-2s} = \dfrac{A}{s} + \dfrac{B}{s-2}$

c) Multiply by s(s – 2)

$\therefore 9\ s - 8 \equiv A(s - 2) + B\ (s)$

d) Put s = 0

$\therefore -8 \equiv A(-2) + B\ (0)\ \therefore A = 4$

e) Put s – 2 , i.e. s = 2

$\therefore\ 10 = A\ (\ 0\ ) + B\ (\ 0\ )\ \therefore B = 5$

$\therefore\ F(t) = L^{-1}\left\{ \dfrac{4}{s} + \dfrac{5}{s-2} \right\} = 4 + 5e^{2t}$

Table of inverse transforms

– Standard transforms

f(s)	F(t)
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}
$\frac{n!}{s^{n+1}}$	t^n (n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ (n a positive integer)
$\frac{a}{s^2+a^2}$	sin at
$\frac{s}{s^2+a^2}$	Cos at
$\frac{a}{s^2-a^2}$	Sinh at
$\frac{s}{s^2-a^2}$	Cos ax

Transforms Of Derivatives

Let $F^{/}(t)$ denote the first derivative of F(t) with respect to t , $F^{//}(t)$ denote the second derivative of F(t) with respect to t, etc.

Then $L\{F^{/}(t)\}=\int_0^\infty e^{-st}F^{/}(t)dt$

Integrating By Parts

by definition ,

$$L\{F^{/}(t)\}=\left[e^{-st}F(t)\right]_0^\infty-\int_0^\infty F(t)(-se^{-st})dt$$

when $t\to 0$, $e^{-st}F(t)\to 0$

$$L\{F^{/}(t)\}=-F(0)+s\int_0^\infty e^{-st}F(t)dt$$

$$L\{F^{/}(t)\}=-F(0)+sL\{F(t)\}$$

$$L\{F^{//}(t)\}=-F^{/}(0)+sL\{F^{/}(t)\}=-F^{/}(0)+s\left[-F(0)+sL\{F(t)\}\right]$$

$$L\{F^{//}(t)\} = s^2 L\{F(t)\} - sF(0) + F^{/}(0)$$

$$L\{F^{///}(t)\} = s^3 L\{F(t)\} - s^2 F(0) - sF^{/}(0) - F^{//}(0)$$

$$L\{F^{iv}(t)\} = s^4 L\{F(t)\} - s^3 F(0) - s^2 F^{/}(0) - sF^{//}(0) - F^{///}(0)$$

Differential Equation And Its Solution

$$\frac{dx}{dt} - 2x = 4 \quad (1)$$

Its Solution is x = -2 + $3e^{2t}$

, To verify it we find $\frac{dx}{dt}$

$$\frac{dx}{dt} = 6e^{2t}$$

then

$$\frac{dx}{dt} - 2x = 6e^{2t} - 2(-2 + 3e^{2t})$$

$$= 6e^{2t} + 4 - 6e^{2t} = 4$$

So equation (1) is satisfied. Hence x = -2 + $3e^{2t}$

is solution of

$$\frac{dx}{dt} - 2x = 4$$

Example Solve the differential equation Taking Laplace transform as

$\frac{dx}{dt} - 2x = 4$ given that at t = 0 , x = 1

$$L\left[\frac{d}{dt}(x(t))\right] - 2L(x(t)) = L(4) \Rightarrow L\left[\frac{d}{dt}(x(t))\right] - 2L(x(t)) = L(4) \Rightarrow sL(x(t)) - x(0) - 2L(x(t)) = \frac{4}{s}$$

$$(s-2)L(x(t)) - x(0) = \frac{4}{s} \Rightarrow (s-2)L(x(t)) - 1 = \frac{4}{s} \Rightarrow (s-2)L(x(t)) = \frac{4}{s} + 1 \Rightarrow (s-2)L(x(t)) = \frac{4+s}{s}$$

$$\Rightarrow L(x(t)) = \frac{4+s}{s(s-2)}$$

$$\Rightarrow x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] \text{------------------------(1)}$$

First we do the partial fraction of $\frac{4+s}{s(s-2)}$

$$\frac{4+s}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)}$$

$\Rightarrow 4 + s = A(s-2) + B(s)$ (2)

Put s = 0 in equation (2) ; 4 = -2A ; A = -

2 Put s = 2 in equation (2) ; 6 = B(2) ;

B = 3

$$So \quad \frac{4+s}{s(s-2)} = \frac{-2}{s} + \frac{3}{s-2}$$

Equation # (1) becomes

$$x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] = L^{-1}\left[\frac{-2}{s}\right] + L^{-1}\left[\frac{3}{s-2}\right]$$

$$= -2 + 3e^{2t}$$

Solution of differential equation by laplace transforms

To solve a differential equation by Laplace transforms, we go through Laplace transforms, we go through four distinct stages.

(a) Re- write the equation in term of Laplace transforms.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

Solve the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$$

given that at t = 0 , x = 5 and

$$\frac{dx}{dt} = 7$$

$x^{//}(t) - 3x^{/}(t) + 2x(t) = 2e^{3t}$ Given x (0) = 5 , x (0) = 7

$$L\left(x^{//}(t)\right) - 3L(x^{/}(t)) + 2L(x(t)) = 2L(e^{3t})$$

$$s^2 L\{x(t)\} - s\,x(0) - x^{/}(0) - 3\{s\,L(x(t)) - x(0)\} + 2L\,x(t) = \frac{2}{s-3}$$

We rewrite the equation in term of its transforms.

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{dx}{dt}\right] + 2L[x] = 2L\left[e^{3t}\right]$$

$$[s^2L(x(t)) - s\,x(0) - x^{/}(0)] - 3[s\,L(x(t)) - x(0)] + 2L(e^{3t})]$$

At t = 0 , x = 5 ,

$$\frac{dx}{dt} = 7$$

So x (0) = 5 , x (0) = 7

$$s^2L(x(t)) - s(5) - 7 - 3\{sL(x(t)) + 3(5)\} + 2L(x(t)) = \frac{2}{s-3}$$

$$s^2L(x(t)) - 3sL(x(t)) + 2L(x(t)) = \frac{2}{s-3} - 8 + 5s$$

$$(s^2 - 3s + 2)L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{s-3}$$

$$L(x(t)) = \qquad =$$

$5s^2 - 23s + 24$

$(s-1)(s-2)(s-3) \quad (s-1)(s-2)(s-3)$

Making Partial fraction of R.H.S , We have

$$L(x(t)) = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

After solving these we get A= 3 , B = 2 and C = 0

So $L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)} + \frac{0}{(s-3)}$

$$L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)}$$

$$x(t) = L^{-1}\{\frac{3}{(s-1)}\} + L^{-1}\{\frac{2}{(s-2)}\}$$

$$= 3e^{t} + 2e^{2t}$$